Solution to Assignment 5

Supplementary Problems

1. Let
$$F(t) = \iiint_B f(x^2 + y^2 + z^2)\, dV \ ,$$
where B is the ball of radius t centered at the origin and f is continuous. Show that $F'(t) = 4\pi t^2 f(t^2)$.

   Solution. In spherical coordinates,
$$F(t) = \int_0^{2\pi} \int_0^{\pi} \int_0^t f(\rho^2)\rho^2 \sin\varphi \, d\rho d\varphi d\theta \ .$$

   Therefore,
$$F(t) = 2\pi \times 2 \times \int_0^t f(\rho^2)\rho^2 \, d\rho \ ,$$

   and
$$F'(t) = 4\pi t^2 f(t^2) \ .$$

Assignment 5 Solution

Please submit the following questions by **23 Feb 2021, 23:00**.

§15.7: Q12, **Q16: duplicated**, Q66

Q12

Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up the triple integrals on cylindrical coordinates that gives the volume of D using the following orders of integration.

(a) $dz\,dr\,d\theta$

(b) $dr\,dz\,d\theta$

(c) $d\theta\,dz\,dr$

Solution:

(a)

$$\int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r\,dz\,dr\,d\theta$$

(b)

$$\int_0^{2\pi} \int_0^1 \int_0^z r\,dr\,dz\,d\theta + \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{2-z}} r\,dr\,dz\,d\theta$$

(c)

$$\int_0^1 \int_r^{2-r^2} \int_0^{2\pi} r\,d\theta\,dz\,dr$$

§15.7 Q16 (Duplicated)

Solution: See HW4 Solution.

Q66

Find the average value of the function $f(\rho, \phi, \theta) = \rho \cos\phi$ over the solid upper ball $\rho \leq 1$, $0 \leq \phi \leq \pi/2$.

Solution: Note that the upper ball has volume $\frac{2}{3}\pi$, hence

$$\text{Average value} = \frac{3}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^3 \cos\phi \sin\phi\,d\rho\,d\phi\,d\theta = \frac{3}{8}$$